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The Cometary Cloud in the Solar System and the Résibois–Prigogine Singular Invariants of Motion

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A relation between nonintegrability of nonlinear dynamical systems with a continuous Fourier spectrum and irreversibility is investigated in terms of the Lie-algebraic formalism. Résibois and Prigogine's singular invariants of motion play an essential role. As an application of the formalism, we solve the restricted three-body problem for the case of nearly parabolic motion of the third body. This gives a model of the motion of a comet in the solar system. The results indicate that there is (deterministic) chaos in the motion of a comet in a nearly parabolic orbit. A possible physical implication of the chaotic motion is the existence of a cometary cloud surrounding the solar system. The theoretical results are compared with numerical results, and show good agreement.

KEY WORDS: Nonintegrability; irreversibility; singular invariants; restricted three-body problem; deterministic chaos; cometary cloud.

1. INTRODUCTION

Among the many fruitful contributions of I. Prigogine to science, I am especially interested in those aspects of his work related to the problem of irreversibility of large systems and its relation to the fundamental structure of dynamics.[1,2] The remarkable progress in nonlinear dynamics over the last decades reveals more and more the importance of his basic concepts in kinetic theory, such as the dissipativity condition and the singular invariant of motion.[3]

In this article, I will show that these basic concepts lead to an extension of Poincaré's theorem concerning integrability for nonlinear dynamical systems with continuous Fourier spectrum and to a natural classification of these systems.

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I then apply the singular invariant to solve a genuine problem of classical mechanics, the restricted three-body problem, for the case of nearly parabolic motion of the third body. We may regard the two primaries of the problem as the sun and Jupiter, and the third body as a comet. In this example, I will show how the idea of the singular invariant solves the difficulty of the "small denominator" due to the resonance effect. This difficulty in nonintegrable systems has been of central interest not only in our specific problem, but in all of nonlinear dynamics since the classical work of Poincaré.[4-9] My solution of the restricted three-body problem shows that the motion of a comet in nearly parabolic motion is chaotic even though it is deterministic. The existence of a chaotic region in orbital space suggests the existence of a cometary cloud in the solar system. This cloud might be the Oort cloud that many astronomers think is the source of comets,[10] or some secondary cloud generated by the Oort cloud.

2. SINGULAR INVARIANTS AND INTEGRABILITY

We consider a system with N degrees of freedom, the Hamiltonian of which is given by the Fourier series

$$\begin{aligned} H(\mathbf{q}, \mathbf{p}, \mu) &= H_0(\mathbf{p}) + \mu V(\mathbf{q}, \mathbf{p}) \\ &= H_0(\mathbf{p}) + \mu\, \Delta k \sum_{\mathbf{k}} V_{\mathbf{k}}(\mathbf{p}) \exp(i\mathbf{k}\cdot\mathbf{q}) \end{aligned} \tag{2.1}$$

Here, $\mathbf{q} \equiv (q_1 \cdots q_N)$, etc., $\Delta k \equiv \Delta k_1 \cdots \Delta k_N$, $k_i = n_i\, \Delta k_i$ with integer n_i, and $\mathbf{k}\cdot\mathbf{q} \equiv k_1 q_1 + \cdots + k_N q_N$. The constant μ is a small perturbation parameter. In other words, the perturbation term depends periodically on q_i with period $2\pi/\Delta k_i$ for $i = 1, \ldots, N$. The discreteness Δk_i of the Fourier spectrum generally depends on the momentum $\mathbf{p}$. Some of the Fourier spectra can be continuous as $\Delta k_i \to 0$ for certain values of $\mathbf{p}$. I am especially interested in this limiting case.

With a systematic use of the Lie-algebraic formalism with the canonical transformation operator U_F, we can construct a formal solution of the equation of motion to any order of the perturbation series.[11,12] This operator is defined as a solution of the operator equation

$$-i\frac{\partial U_F}{\partial \mu} = L_F U_F \tag{2.2}$$

with the boundary condition $U_F = 1$ at $\mu = 0$. Here $L_F \equiv i\{F, \cdot\}$ is a Lie derivative generated by a generating function $F(\mathbf{q}, \mathbf{p}, \mu)$ and $\{\ ,\ \}$ is the Poisson bracket with respect to the canonical variables $(\mathbf{q}, \mathbf{p})$. The canonical transformation is given by $q_i' = U_F^{-1} q_i$ and $p_i' = U_F^{-1} p_i$, where $(\mathbf{q}', \mathbf{p}')$ are new canonical variables and the inverse operator U_F^{-1} satisfies the equation $i\, \partial U_F^{-1}/\partial \mu = U_F^{-1} L_F$ with the condition $U_F^{-1} = 1$ at $\mu = 0$.

Suppose that we can find a suitable generating function F such that the operator U_F makes the Hamiltonian cyclic in the form

$$U_F H(\mathbf{q}, \mathbf{p}, \mu) = \tilde{H}(\mathbf{p}, \mu) \tag{2.3}$$

then the new momenta p_i on the right-hand side of this expression are invariants of the motion. In the following context we will show with a perturbation analysis that these new momenta reduce to Résibois and Prigogine's singular invariants in the limiting case of the continuous Fourier spectrum.

We expand the generating function, the transformation operator, and the new Hamiltonian in power series of μ,

$$F = \sum_{n=0}^{\infty} \mu^n F_{n+1}, \qquad U_F = \sum_{n=0}^{\infty} \mu^n U_n, \qquad \tilde{H} = \sum_{n=0}^{\infty} \mu^n \tilde{H}_n \tag{2.4}$$

Then Eq. (2.2) gives a recursive relation with $U_0 = 1$

$$U_n = \frac{1}{n} \sum_{m=0}^{n-1} i L_{n-m} U_m \tag{2.5}$$

where $L_m = L_{F_m}$. Combining Eq. (2.5) with Eq. (2.3) gives

$$\begin{aligned} H_0 &= \tilde{H}_0 \\ i L_1 H_0 + V &= \tilde{H}_1 \\ \tfrac{1}{2}[(iL_1)^2 + iL_2]\, H_0 + iL_1 V &= \tilde{H}_2 \end{aligned} \tag{2.6}$$

and so on. By solving Eq. (2.6) step by step, we obtain F_n for $n = 1, 2, 3, \ldots$. This gives us the desired U_F in Eq. (2.3).

Using a Fourier analysis, we have the first-order approximation, for example, $\tilde{H}_1(p) = \Delta k V_0(\mathbf{p})$ and

$$F_1(\mathbf{q}, \mathbf{p}) = \Delta k \sum_{\mathbf{k}}{}' \frac{-i V_{\mathbf{k}}(\mathbf{p})}{\mathbf{k} \cdot \boldsymbol{\omega}} \exp(i \mathbf{k} \cdot \mathbf{q}) \tag{2.7}$$

here the prime on the summation sign stands for $\mathbf{k} \neq 0$, and $\boldsymbol{\omega} \equiv \partial H_0 / \partial \mathbf{p}$.

This solution is, however, still formal and has a possibility of divergence in the general case because Eq. (2.7) includes the "small denominator" $\mathbf{k} \cdot \boldsymbol{\omega}$. Note that if we consider the case of the continuous limit $\Delta k \to 0$, then Eq. (2.7) reduces to the Fourier integral. The resulting integration is a Cauchy integral evaluated on the real axis of k_i. Hence, the integral is well-defined and has a finite discontinuity on the real axis for an appropriate potential such as the example of the restricted three-body problem discussed later.

The new momentum p_i' is thus given for the continuous limit by

$$p_i' = p_i + \mu\, \Delta k \sum_{\mathbf{k}} \frac{k_i V_{\mathbf{k}}(\mathbf{p})}{\mathbf{k} \cdot \boldsymbol{\omega} - i\varepsilon} \exp(i\mathbf{k} \cdot \mathbf{q}) + O(\mu^2) \tag{2.8}$$

where we have used the conventional notation $\Delta k_i \sum_{k_i}$ for the integral notation $\int_{-\infty}^{+\infty} dk_i$ for the continuous Fourier component. Here ε is a positive infinitesimal and we have chosen a branch of the analytic continuation that is consistent with the boundary condition such that the old momentum $\mathbf{p}(t)$ reduces to the new momentum $\mathbf{p}'$ (which is *constant* in time) in the limit of $t \to -\infty$.

The new momentum does not reduce to a collisional invariant[1,13] such as the unperturbed Hamiltonian in the limit of $\mu \to 0$. Hence, our new momentum is just the singular invariant introduced by Résibois and Prigogine.[3] The important consequence of the singular invariant is that we can evaluate the resonance effect to construct the solution for the equation of motion. [Note that the residue of the integration in Eq. (2.8) is evaluated just at the resonance point $\mathbf{k} \cdot \boldsymbol{\omega} = 0$.] This raises the hope of avoiding Poincaré's catastrophe[1,4] for the perturbation analysis in nonlinear dynamical systems with the continuous Fourier spectrum.

I have recently investigated more closely the relation between the Lie-algebraic formalism and Poincaré's theorem for the case of continuous Fourier spectrum. Here I summarize the results (detailed discussion can be found in Ref. 14): In the perturbation expansion of the transformation operator (2.5) there appear divergent terms of the form

$$\psi(+i\varepsilon)/\varepsilon \qquad (\varepsilon \to 0+) \tag{2.9}$$

Here, ψ is the collision operator, which is a basic quantity in modern kinetic theory,[2,15] and is given by

$$\psi(z) = \mu^2 P L_V \frac{1}{L_{H_0} - z} L_V P + O(\mu^3) \tag{2.10}$$

where L_{H_0} and L_V are the Lie derivatives generated by H_0 and V, respectively, and P is the projection operator, which projects out the $\mathbf{q}$-average components, i.e., the $f_{\mathbf{k}}(\mathbf{p})$ component with $\mathbf{k} = 0$ in the Fourier representation, of any phase function $f(\mathbf{q}, \mathbf{p})$. As a consequence, we encounter Poincaré's results, even in the case of the continuous spectrum: whenever the collision operator has to be retained, the dynamical system cannot be integrable. Except for trivial invariants of motion, such as the total Hamiltonian, general invariants can only be constructed if the collision operator $\psi(+i\varepsilon)$ vanishes.

If we recall the fact that the existence of the collision operator gives a criterion of the existence of an irreversible process,[1] the above result leads to a remarkable relation between the integrability and the irreversibility of a dynamical system: the irreversible behavior of the system is a direct consequence of Poincaré's catastrophe.

A typical example of the case where $\psi(+i\varepsilon)$ does not vanish is a gaseous system. In this case a dominant term of the collision operator is proportional to a small factor $\Delta k \sim 1/\Omega$ and large factor N with $N/\Omega =$ finite for the thermodynamic limit, where Ω is the volume of the system and N is the number of particles.[1]

On the other hand, a typical example of the case where $\psi(+i\varepsilon)$ vanishes is a restricted three-body problem with nearly parabolic orbit for the third body, which we discuss in the next section. In this case the collision operator is proportional to a vanishing factor $\Delta k \sim 1/T$, where T is the period of an elliptic motion and $T \to \infty$ in the parabolic limit, while there is no large factor such as N in the case of the gaseous system mentioned above. Therefore, in this case we can apply the perturbation analysis to construct the solution of the motion.

3. APPLICATION: THE COMETARY CLOUD

We apply the above argument to construct a solution of the two-dimensional circular restricted problem in the case of nearly parabolic motion of the third body.[16] We may regard the two primaries as the sun and Jupiter, and the third body as a comet.

To describe the motion, we use the following dimensionless units: the distance between the two primaries, their total mass, and their angular velocity are all normalized to 1. By using canonical variables (g, G) and (Q, P) which are closely related to Delaunay's variables,[5,17] we can write Hamiltonian for the direct motion of the comet in the synodic coordinate system as

$$H = \tfrac{1}{2}P - G + \mu V \tag{3.1}$$

where

$$\mu V = \frac{1}{r} - \frac{1}{r_1} + \mu\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \tag{3.2}$$

Here, g is the argument of the perihelion measured in the rotating system, G is the angular momentum, and $P \equiv 1/a$, where a is the semimajor axis of the conic motion in the two-body problem; $P>0$, $P=0$, and $P<0$ correspond to hyperbolic, parabolic, and elliptic motion, respectively; and Q is a canonical conjugate coordinate of P. Jupiter's mass μ is given by $\mu = 0.95\times 10^{-3}$ in our units. The distances r_1 and r_2 are given by the synodic polar coordinate system (r, ϑ) measured from the center of mass of the two primaries such that

$$\begin{aligned} r_1 &= (r^2 - 2\mu r\cos\vartheta + \mu^2)^{1/2} \\ r_2 &= [r^2 + 2(1-\mu)r\cos\vartheta + (1-\mu)^2]^{1/2} \end{aligned} \tag{3.3}$$

The caonical transformation between our canonical variables and the polar coordinate system $(r, \vartheta, p_r, p_\vartheta)$ is given by

$$\begin{aligned} p_r &= \pm\left(-\frac{G^2}{r^2}+\frac{2}{r}+P\right)^{1/2}, \qquad p_\vartheta = G \\ Q &= \frac{1}{2P^{3/2}}(e\sinh u - u), \qquad g = \vartheta - \arccos\frac{1}{e}\left(\frac{G^2}{r}-1\right) \end{aligned} \tag{3.4}$$

with $u \equiv [2(e-1)]^{1/2}\tau$. Here $e \equiv (1+PG^2)^{1/2}$ is the eccentricity for the case of the two-body problem. The auxiliary variables u and τ are related to r through the relation $r = a(e\cosh u - 1)$.

In contrast to Delaunay's variables, our variables (Q, g, P, G) are well-defined and continuous in the limiting case of the parabolic motion, i.e., $|P|\to 0$ and $e\to 1$ with $q\equiv a(e-1) = \text{finite}$, where q is the perihelion distance of the comet for $\mu=0$. In this limit, we have the simple relations

$$Q = (q^3/2)^{1/2}(\tau + \tau^3/3), \qquad r = q(1+\tau^2) \tag{3.5}$$

If we restrict ourselves to the case of $q>1$, we can expand the potential (3.2) in terms of Legendre polynomials. Keeping the lowest order terms in μ, we obtain

$$\begin{aligned} \mu V = {} & \frac{\mu}{4r^3}\left(1+\frac{9}{16r^2}+\cdots\right) - \frac{3\mu}{8r^4}\left(1+\frac{10}{16r^2}+\cdots\right)\cos\vartheta \\ & + \frac{\mu}{4r^3}\left(3+\frac{20}{16r^2}+\cdots\right)\cos 2\vartheta + \cdots \end{aligned} \tag{3.6}$$

Substituting Eq. (3.5) for nearly parabolic motion in Eq. (3.6) and representing the function of Q in Fourier expansion, we obtain

$$\mu V = \mu\,\Delta k_1 \sum_{k_1}\left[v_{k_1}^0 + \sum_{j=1}^{\infty}(v_{k_1}^{cj}\cos jg + v_{k_1}^{sj}\sin jg)\right]\exp(ik_1 Q) \tag{3.7}$$

where $k_1 = n\,\Delta k_1$ with integer n. For the elliptic case, $\Delta k_1 = 2(-P)^{3/2}$ and the Fourier spectrum is discrete. For the parabolic and hyperbolic cases $\Delta k_1 \to 0$ and the spectrum is continuous.

The explicit forms of the Fourier components V_{k_1,k_2} can be calculated by using (3.5) in nearly parabolic orbits. The resulting expressions are an intricate series of modified Bessel function and incomplete gamma function. A detailed calculation of the Fourier components is found in Ref. 18. The result is as follows: The solution of $P(t)$ to the first-order approximation is

$$P(t) = U_F P_0 = P_0 - \mu \sum_{k_2=-1}^{1}{}' \int_{-\infty}^{+\infty} dk_1 \frac{k_1 V_{k_1 k_2}(P_0, G_0)}{k_1\omega_1 + k_2\omega_2 - i\varepsilon} e^{i[k_1 Q(t) + k_2 g(t)]} \tag{3.8}$$

where $Q(t) = \omega_1(t - t_0)$ and $g(t) = \omega_2(t - t_0) + g_0$, with $\omega_1 = 1/2$ and $\omega_2 = -1$. For sufficiently large positive time, we can evaluate the integral in k_1 in Eq. (3.8) in such a way that $P(t)$ approaches asymptotically to P_{as}, where

$$P_{\text{as}} = P_0 + 2\mu \sum_{j=1}^{\infty} \Delta_j \sin jg_0 \tag{3.9}$$

where $\Delta_j \equiv 2\pi j(v_{2j}^{cj} - iv_{2j}^{sj})$, and the angle g_0 is the phase angle of Jupiter measured from the direction of the unperturbed perihelion of the comet when the comet passes the perihelion. Note that $P(t)$ asymptotically does not depend on time. For moderately large q, we obtain

$$\Delta_1 = [2^{1/4}\sqrt{\pi}\, q^{-1/4} + O(q^{-1})] \exp(-4q^{3/2}/3\sqrt{2}) \tag{3.10}$$

and $\Delta_j \sim \exp(-4jq^{3/2}/3\sqrt{2}) \ll 1$ for $j \geqslant 2$. Hence, we may neglect the contribution from terms with $j \geqslant 2$ in Eq. (3.9).

Figure 1 shows the q dependence of the amplitude of $(P_{\text{as}} - P_0)/2$, which is estimated theoretically by $\mu\Delta_1$ and also evaluated numerically based upon the Runge–Kutta integration method performed with the original rectangular coordinates.[2] In order to improve the precision of the numerical integration, I amplified the effect of Jupiter by using $\mu = 10^{-2}$ instead of $\mu = 0.95 \times 10^{-3}$. The figure shows good agreement between the theory and the numerical simulation.

So far we have studied a single scattering process of a comet by the effect of the sun and Jupiter. We now study multiple scattering process as applied to the capture of comets in the solar system. The orbit of the comet

[2] The numerical simulation was done by Dr. R. Broucke.

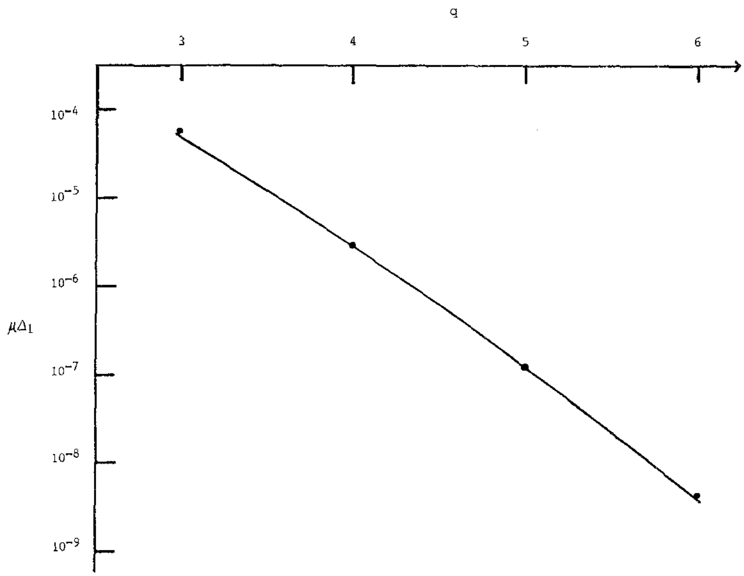

Fig. 1. Dependence of $(P_{as} - P_0)/2$ on the perihelion distance q ($\mu = 0.01$). (—) Theoretical result. (●) Numerical result.

is elliptic if the asymptotic value of P_{as} is negative. Its period is given approximately by $2\pi/(-P)^{3/2}$ in accordance with Kepler's third law. Therefore, we can calculate the new phase angle of Jupiter when the comet once again passes the perihelion. At each passage, the perihelion distance q also changes, because of the relation $G = [q(e+1)]^{1/2} \simeq (2q)^{1/2}$. By the same operation used in deriving Eq. (3.9), it is easy to see that q changes in each scattering process with order of $\mu\, \Delta_1(q_0)$, where q_0 is the initial value of q. Therefore, in accordance with our approximation of neglecting the effects of $\mu\Delta_j$ for $j \geqslant 2$, we can set $\mu\, \Delta_1(q)$ equal to its initial value $\mu\, \Delta_1(q_0)$ for each passage. Combining this argument with Kepler's third law, we obtain a canonical (area-preserving) map describing multiple scattering process of the comet, which I call the "Keplerian map,"

$$
\begin{aligned}
P_{n+1} &= P_n + 2\mu\, \Delta_1(q_0) \sin g_n \\
g_{n+1} &= g_n - 2\pi/(-P_{n+1})^{3/2}; \qquad (\mathrm{mod}\ 2\pi)
\end{aligned}
\tag{3.11}
$$

For a given initial condition (g_0, P_0), if, for a certain value of n, P_n becomes positive, the comet escapes from the solar system. Figure 2 shows a distribution of points in the (g, P) space obtained by iterations of the Keplerian map for several initial conditions. This figure shows that there are chaotic motions in the neighborhood of the parabolic orbits. In the P

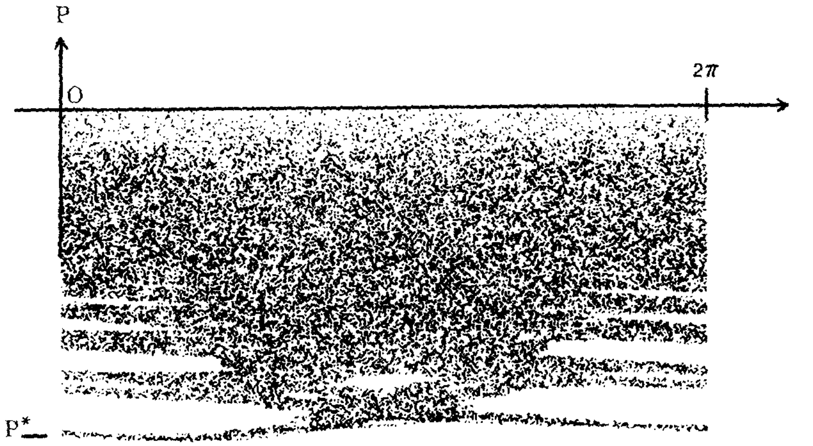

Fig. 2. A distribution of points in the (g, P) space obtained by iteration of the Keplerian map, Eq. (4.1), for $\mu\Delta_1 = 0.01$. The calculations were performed with the accuracy of 16 digits. P^* is the lower limit of the chaotic motion, related to the breakdown of the invariant curve in the KAM theory.

space the chaotic region is confined from below by the KAM torus.[7–9,19] There are several regular regions around the elliptic fixed points, and we see the formation of island structure in the regular regions, inside the chaotic region.

The threshold value of the chaotic region in P space is estimated by reducing the Keplerian map to the "standard map" around fixed solutions of Eq. (3.11).[19] This gives $P^* = -(6\pi\mu\Delta_1/K_c)^{2/5}$, where K_c is a critical value of the parameter in the standard map given by $K_c = 0.9716$.[20] The theoretical value P^* is indicated in Fig. 2.

From this figure, it is clear that a simple application of the stochastic description such as the diffusion equation to describe the cometary motion, which has been done by Lyttleton and Hammersley[21] (see also Ref. 10), might fail near the boundary region of the chaotic motion.

The theoretical value of P^* gives us the order of the scale of the semimajor axis $a^* = 1/P^*$ for the innermost orbit in the chaotic region. For example, we have $a^* \sim 10^4$ A.U. for $q = 6$.

Suppose that the points in Fig. 2 indicate an ensemble of the comets instead of a history of a time sequence of a few comets. Then, this figure suggests the existence of a cometary cloud such as the Oort cloud that is thought of as the source of comets.[10] I hope this formulation reveals a detailed dynamical structure of the cometary cloud.

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